Numerical Linear Algebra

Mid-term test

Fall 2015

Variant 1

Questions (40 pts)

Q1 1.1 What is the complexity of naive matrix-matrix multiplication? (5 pts)

1.2 Is it possible to reduce this complexity for general matrices? If so, give example of an algorithm. (5 pts)

Q2 Suppose that you want to calculate the best low-rank approximation of a certain matrix with relative precision ϵ. What should you do? How much memory do you need to store the full matrix and the rank-r approximation? (10 pts)

Q3 What method for finding the largest eigenvalue do you know? When does it fail to converge? (10 pts)

Q4 Formulate the least squares problem for overdetermined systems. What are the ways to solve it? (10 pts)

Problems (60 pts)

P1 Find $\|F_n\|_\infty$ and $\|F_n\|_2$, where F_n is the Fourier matrix of size $n \times n$. (10 pts)

P2 Find the third singular value $\sigma_3(A)$ of matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. (10 pts)

P3 Let $D = \mathrm{diag}(d_1, \dots, d_n, 0, \dots, 0)$, $d_1, \dots, d_n \neq 0$. Prove, that for its pseudoinverse holds $D^\dagger = \mathrm{diag}\left(\frac{1}{d_1}, \dots, \frac{1}{d_n}, 0, \dots, 0\right)$. (10 pts)

P4 Show that any normal triangular matrix is diagonal. (10 pts)

P5 Find $\mathrm{cond}_2 \begin{bmatrix} \epsilon^2 & 0 \\ 0 & \epsilon \end{bmatrix}$, where $\epsilon \neq 0$. Note: subscript 2 in cond_2 means that second norm is used. (10 pts)

P6 The goal of compressed sensing is to find the sparsest solution x of an undetermined linear system $y = Ax$ where $A \in \mathbb{R}^{n \times m}$, $n < m$. In order to achieve it one could try to find solution which has minimal first norm. Intuition behind this fact is quite simple in 2D:

6.1 Draw disks $\|x\| = \text{const}$ in $1, 2$ and ∞ norms. (5 pts)

6.2 Find graphically solutions of $y = Ax$, $\|x\|_* \to \min$, where $A \in \mathbb{R}^{1\times 2}$ and $* = \{1, 2, \infty\}$. Which norm yields the sparsest solution? (5 pts)